MATH 256

2. First Order ODE

Integrating Factor

$$\frac{dy}{dt} + p(t)y = g(t)$$

Sol'n:

$$y = \frac{1}{N}\left(c + \int Ng\right)$$

where:

$$N = e^{\int p\,dt}$$

Homogenous ODE

$$\frac{dy}{dx} = f\left(\frac{x}{y}\right)$$

ie: $\frac{dy}{dx} = \frac{y^2}{x^2+y^2}$

Sol'n:
replace y with xV
$y = xV$

Now $\frac{dv}{dx} = \frac{x^2v^2}{x^2+x^2v^2}$

$\frac{dv}{dx} = \frac{v^2}{1+v^2}$

then:
$xV' = \frac{v^2}{1+v^2} - V$ SEPERABLE

Solve this

$$\frac{(1+v^2)\,dv}{(v^2-v-v^3)} = \frac{1}{x}dx$$

Possible ODE's (1st order)

1. $\frac{dy}{dt} + PY = g$ | Integrating factor
2. $\frac{dy}{dt} + PY = gY^n$ | Bernoulli's
3. $\frac{dy}{dx} = f(x)g(y)$ | Seperable
4. $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ | Homogenous
5. $\frac{dy}{dt} = y^2$ | Non-Linear

Bernoulli's eqn

$$\frac{dy}{dt} + p(t)y = g(t)y^n$$

Sol'n
solve $v' + (1-n)pV = (1-n)g$ *can use integrating method

where:
$V = y^{1-n}$

Interval of Existence

- get restrictions on variable.
ie: $x \neq \pm 1$, then start at IC
ie: IC $= 0$. then interval is

$$-1 < x < 1$$

if IC $= 5$, then $1 < x < \infty$

Applications

Bank:

$$\frac{ds}{dt} - rs = k$$

where:
$p = -r$ $\quad N = e^{\int p} = e^{\int r\,dt}$
$g = k$

Sol'n:

$$S = Ce^{rt} - \frac{k}{r}$$

$$= \left(S_0 + \frac{k}{r}\right)e^{rt} - \frac{k}{r}$$

Seperable ODE

$$\frac{dy}{dx} = f(y)\,g(x)$$

Sol'n:

$$\frac{dy}{f(y)} = g(x)dx$$

$$\int \frac{1}{f(y)}dy = \int g(x)dx$$

Non-Linear ODE

ex: $y' = \frac{t^2}{y^2-1}$; $y(0) = 1$

what determines interval:
1. IC
2. Equation
3. Solution

Critical Points

$$\frac{dy}{dx} = f(y)$$

crit points where $\rightarrow f(y) = 0$

Case 1 $f'(y_0) < 0$ Stable
Case 2 $f'(y_0) > 0$ unstable

Autonomas ODE and Population

Model 1 Linear

$$y' = ry$$

Model 2 Logistic

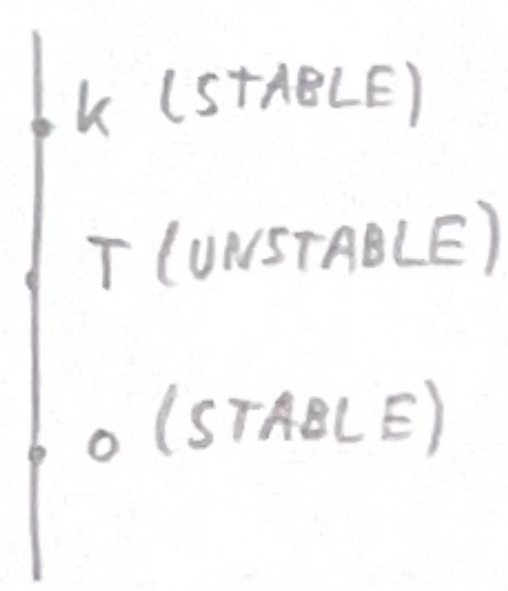
$$y' = ry\left(1 - \frac{y}{k}\right)$$

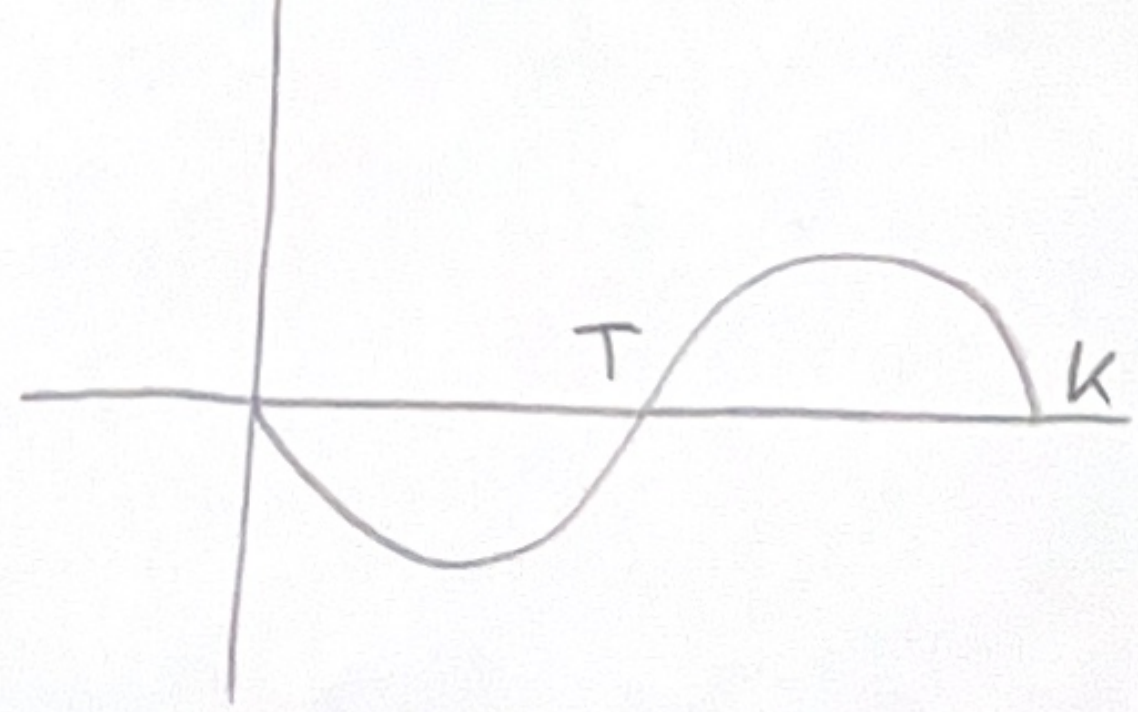
Model 3 Threshold

$$y' = -ry\left(1 - \frac{y}{T}\right)$$

Model 4 Logistic with Threshold

$$y' = -ry\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{k}\right) \quad 0 < T < k$$





3 Second Order ODE

Standard Form

$y'' + p(t)y' + q(t)y = g(t)$

Wronskian

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

if $W \neq 0$, y_1, y_2 are fundamental set of solutions

$W = e^{-\int p\,dt}$

Abel's formula:

$W' + pW = 0$

If one is true, all are true;

- $\{y_1, y_2\}$ F.S.S
- $W \neq 0$
- any sol'n can be written as $ay_1 + by_2$

Case 1 Constant Coefficients

$ay'' + by' + cy = 0$

$ar^2 + br + c = 0$

$y_1 = c_1 e^{r_1 t}$

$y_2 = c_2 e^{r_2 t}$

$y = ae^{r_1 t} + be^{r_2 t}$

Case 2 Complex Roots

$ay'' + by' + cy = 0$

$ar^2 + br + c = 0$

$y_1 = c_1 e^{(\lambda + iN)t}$

$y_2 = c_2 e^{(\lambda - iN)t}$

$y = c_1 e^{\lambda t}\cos(Nt) + c_2 e^{\lambda t}\sin(Nt)$

Case 3 Repeated Roots

$ay'' + by' + cy = 0$

$ar^2 + br + c = 0$

$* r_1 = r_2$

$y = c_1 e^{rt} + c_2 t e^{rt}$

$y_1 = c_1 e^{rt}$

$y_2 = c_2 t e^{rt}$

Case 4 Reduction of order

"Non constant coefficients, but we know y_1"

$W = e^{-\int p\,dt}$

$y_2 = y_1 \int \frac{W}{y_1^2}\,dt$

$y = c_1 y_1 + c_2 y_2$

Case 5 Euler Type

$t^2 y'' + 3ty' + y = 0$

$at^2 y'' + bty' + cy = 0$

$ar(r-1) + br + c = 0$

$y_1 = c_1 e^{r_1 t}$

$y_2 = c_2 e^{r_2 t}$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Case 6 Non-Homogenous

$y'' + p(t)y' + q(t)y = g(t)$ (1)

$y'' + p(t)y + q(t)y = 0$ (2)

y_1, y_2 are F.S.S to eqn (2)

Y_p is a special solution to eqn(1)

$y(t) = c_1 y_1 + c_2 y_2 + Y_p$

Method of undetermined Coefficients

ex: $y'' - 4y' - 12y = 3e^{5t}$

note: $r_1 = -2$, $r_2 = 6$

guess $Y_p = Ae^{5t}$

$Y_p' = 5Ae^{5t}$

$Y_p'' = 25Ae^{5t}$

$25Ae^{5t} - 4(5Ae^{5t}) - 12(5Ae^{5t}) = 3e^{5t}$

$-7Ae^{5t} = 3e^{5t}$

$-7A = 3$

$\therefore A = \frac{-3}{7}$

$Y_p = \frac{-3}{7}e^{5t}$

$y(t) = c_1 e^{-2t} + c_2 e^{6t} - \frac{3}{7}e^{5t}$

Method of Variation of Parameters

$Y_h = c_1 y_1 + c_2 y_2$

$Y_p = u_1 y_1 + u_2 y_2$

$W = y_1 y_2' - y_2 y_1' \neq 0$

To solve for u_1, u_2:

$u_1' y_1 + u_2' y_2 = 0$ (1)

$u_1' y_1' + u_2' y_2' = g(t)$ (2)

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

or directly:

$Y_p = -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W}$

where:

$u_1 = -\int \frac{y_2 g}{W}$ $u_2 = \int \frac{y_1 g}{W}$

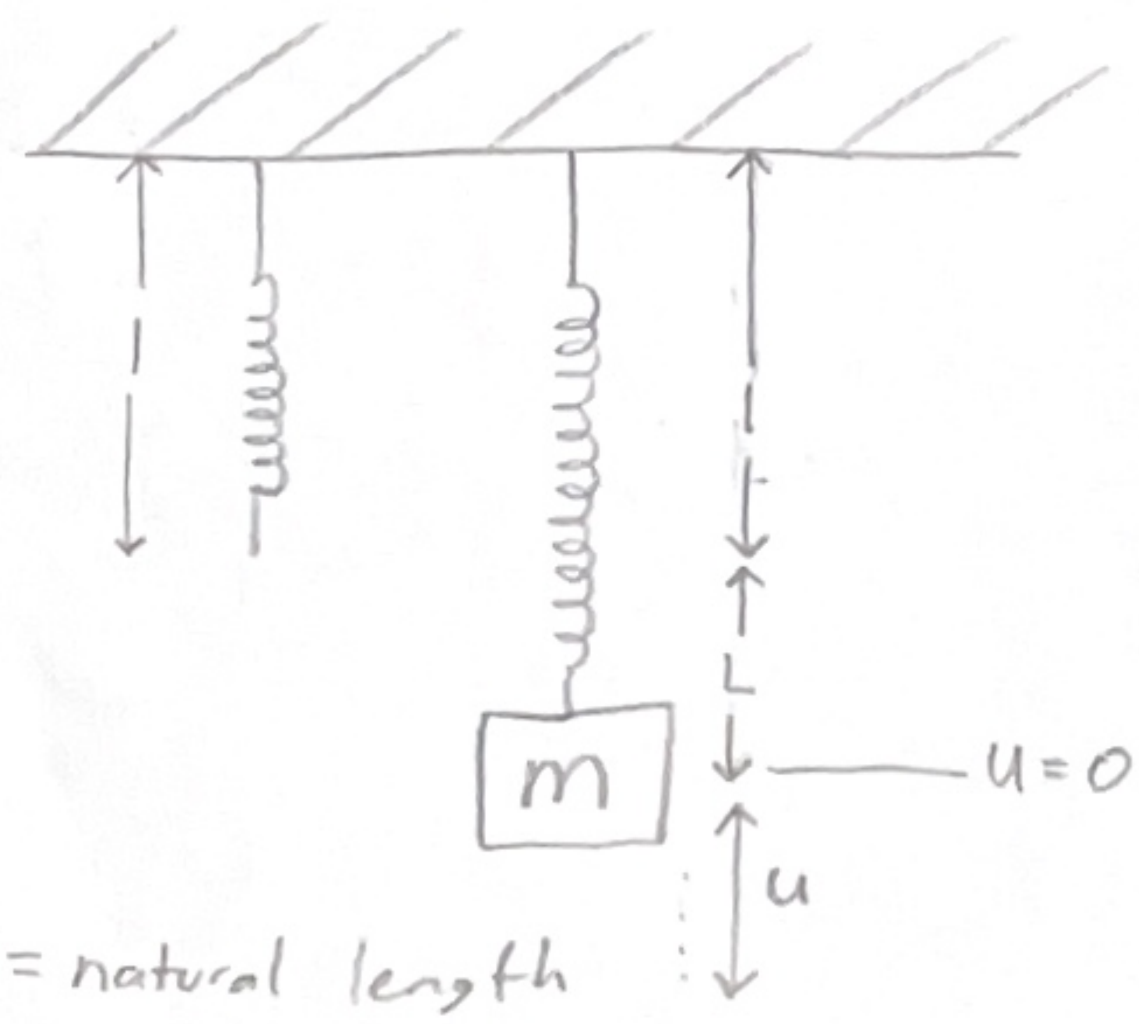
$y(t) = Y_h + Y_p$

$y(t) = c_1 y_1 + c_2 y_2 - y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W}$

Undetermined Coefficients Table

$g(t)$	$Y_p(t)$	Example
$e^{\alpha t}$	$t^s e^{\alpha t}$	$t^s A e^{5t}$
$(\text{Polynomial})^n e^{\alpha t}$	$t^s (\text{Polynomial})^n e^{\alpha t}$	$t^s (At^2+Bt+C) e^{5t}$
$e^{\alpha t}\cos\beta t$ or $e^{\alpha t}\sin\beta t$	$t^s [A\cos\beta t + B\sin\beta t]$	$t^s (A\cos(4x)+B\sin(4x)) e^{6t}$
$(\text{Polynomial})^n e^{\alpha t}\cos\beta t$	$t^s [(\text{Polynomial})^n \cos\beta t + (\text{Polynomial})^n \sin\beta t] e^{\alpha t}$	$t^s [(At^2+Bt+C)\cos(7t) + (Dt+E)\sin(7t)] e^{8t}$

Mechanical Vibrations



l = natural length

L = displacement from l (Equilibrium position)

u = displacement from equilibrium length

Forces

$F_g = mg$

$F_s = -k(L+u)$

$F_d = -\gamma \frac{du}{dt}$ (Friction) (Damping)

Known:

$m = \frac{W}{g}$

$k = \frac{W}{L}$

$W = mg$

* $\boxed{mg = kL}$

Form:

$mu'' + \gamma u' + ku = F_e(t)$; $u(0) = u_0$, $u'(0) = u_1$

Case 1 (Free) $F_e(t) = 0$, (Undamped) $\gamma = 0$

$mu'' + ku = 0$

roots:

$r = \pm i\sqrt{\frac{k}{m}}$

$r = \pm i w_0$

where:

$w_0 = \sqrt{\frac{k}{m}}$ (natural frequency)

$$\boxed{\begin{aligned} u &= A\cos(w_0 t) + B\sin(w_0 t) \\ &= R\cos(w_0 t - \gamma) \end{aligned}}$$

$R = \sqrt{A^2 + B^2}$ $\qquad \gamma = \tan^{-1}\left(\frac{A}{B}\right)$

$\left\{\begin{array}{l} R\cos\gamma = A \\ R\sin\gamma = B \end{array}\right\}$ SOLVE

Case 2 (Free), $F_e = 0$, (Damping) $\gamma > 0$

$mu'' + \gamma u' + ku = 0$

roots:

$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

3 scenarios:

① $\gamma^2 - 4mk > 0$ **OVERDAMPING**

$u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

r_1, r_2 must be negative

so as $t \to \infty$, $u \to 0$

② $\gamma^2 - 4mk = 0$ **CRITICAL DAMPING**

$u(t) = C_1 e^{\frac{-\gamma t}{2m}} + C_2 e^{-\gamma t/2m}$

③ $\gamma^2 - 4mk < 0$ **UNDERDAMPING**

$r_1, r_2 = \lambda \pm Ni$

$u(t) = C_1 e^{\lambda t}\cos(Nt) + C_2 e^{\lambda t}\sin(Nt)$

$= Re^{\lambda t}\cos(Nt - \gamma)$

Case 3 (Forced) $F_e \neq 0$, (Undamped) $\gamma = 0$

$mu'' + ku = F_0\cos(wt)$

2 Scenarios: By undetermined coefficients

① $w_0 \neq w$ $\qquad (-mw^2 A + kA)\cos wt + (-mw^2 B + kB)\sin wt = F_0\cos(wt)$

Solving yields $\rightarrow$ $A = \frac{F_0}{k - mw^2}$

$B = 0$

So $u_h = C_1\cos(w_0 t) + C_2\sin(w_0 t)$

$u_p = \frac{F_0}{m(w_0^2 - w^2)}\cos(wt)$

$$\boxed{\begin{aligned} u(t) &= C_1\cos(w_0 t) + C_2\sin(w_0 t) + \frac{F_0}{m(w_0^2 - w^2)}\cos(wt) \\ u(t) &= R\cos(w_0 t - \gamma) + \frac{F_0}{m(w_0^2 - w^2)}\cos wt \end{aligned}}$$

② $w_0 = w$ (Resonance)

$$\boxed{\begin{aligned} u(t) &= C_1\cos(w_0 t) + C_2\sin(w_0 t) + \frac{F_0}{2mw_0} t\sin(w_0 t) \\ u(t) &= R\cos(w_0 t - \gamma) + \frac{F_0}{2mw_0} t\sin(w_0 t) \end{aligned}}$$

Case 4 (Forced) $F_e \neq 0$, (Damped) $\gamma > 0$

$mu'' + \gamma u' + ku = F(t)$

u_h = Same as **Case 2**

u_p = determined by undetermined coefficients or variation of parameters

Systems of Equations

$A\vec{n} = \lambda \vec{n}$

where λ is an eigenvalue
$\vec{n}$ is the eigenvector

$\det(A - \lambda I) = 0 \rightarrow$ Characteristic Polynomial

Solutions to System

$x' = Ax$

Sol'n $x = \vec{n} e^{\lambda t}$

Case 1: Real Eigenvalues

$x' = (A)x$

sol'n $X = C_1 (n^{(1)}) e^{\lambda_1 t} + C_2 (n^{(2)}) e^{\lambda_2 t}$

Case 2: Complex Eigenvalues

$X_1(t) = \begin{pmatrix} 1+8i \\ 5 \end{pmatrix} e^{(2+8i)t}$

$= e^{2t} [\cos(8t) + i\sin(8t)] \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$

$X_1(t) = \begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} e^{2t} + i \begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix} e^{2t}$

$\therefore X = C_1 \begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix} e^{2t}$

Case 3: Repeated Eigenvalues

$\lambda_1 = \lambda_2 = \lambda$

$X_1 = e^{\lambda t} \vec{n}$

$X_2 = t e^{\lambda t} \vec{n} + e^{\lambda t} \vec{\rho}$

where $\vec{\rho}$ is a sol'n to $(A - \lambda I)\vec{\rho} = \vec{n}$

$\therefore X(t) = C_1 e^{\lambda t} \vec{n} + C_2 (t e^{\lambda t} \vec{n} + e^{\lambda t} \vec{\rho})$

Inhomogenous Systems

Form:

$x' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} x + \begin{pmatrix} 2 \\ -4 \end{pmatrix} t$

new term

Method 1 Undetermined coefficients:

① Solve homogenous System

$\vec{X}_h(t) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}$

② Guess X_p

③ plug X_p into x' and x

④ Solve System

$X_p' = A X_p + t g$ for

$X_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$

Guesses

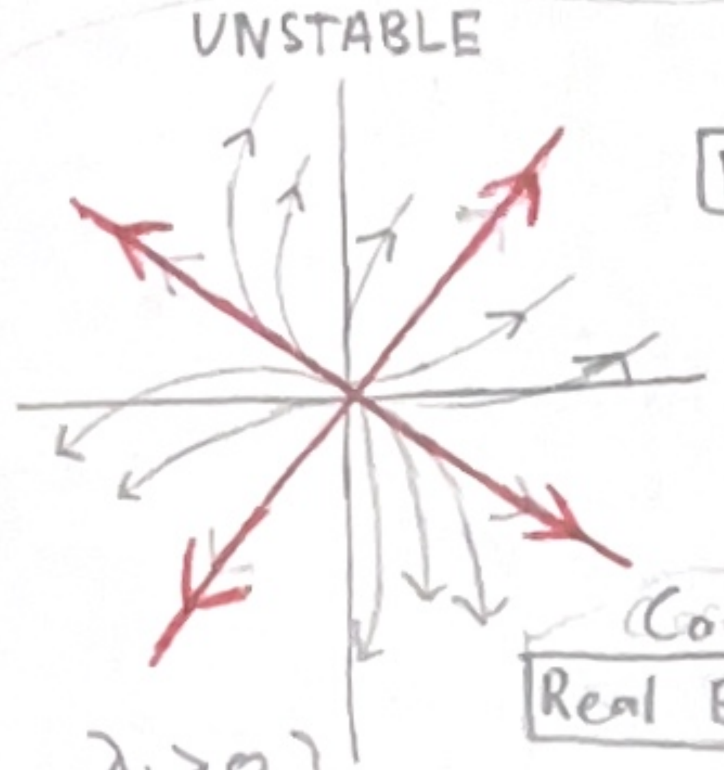
Method 2

Variation of Parameters:

$X_p = \Phi \int \Phi^{-1} g \, dt$

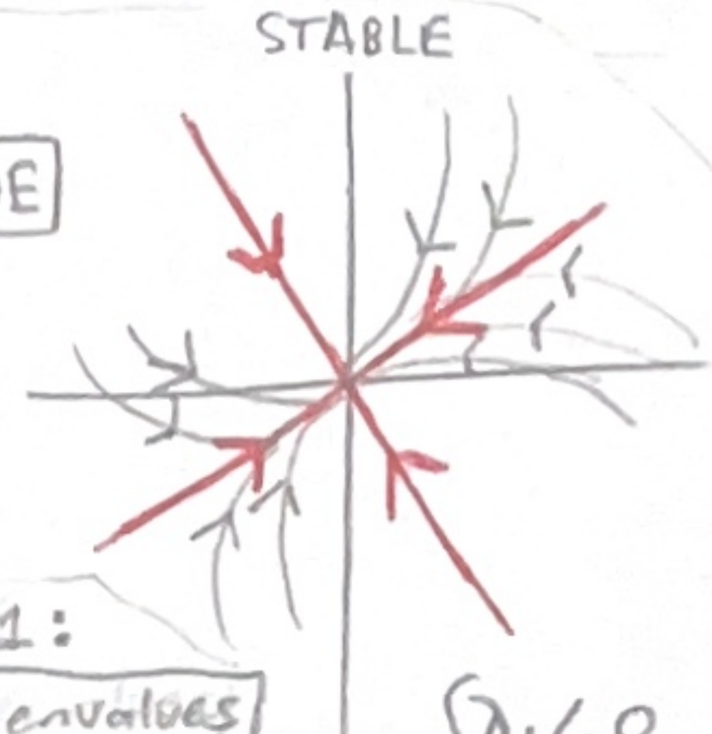
Phase Portraits

UNSTABLE



NODE

STABLE



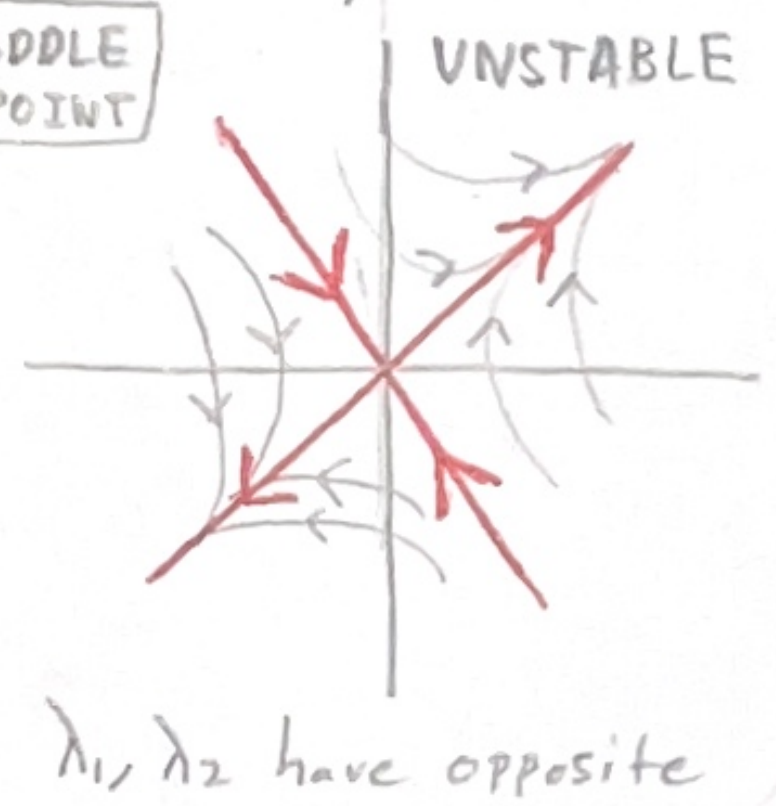
Case 1:
Real Eigenvalues

$\lambda_1 > 0$, $\lambda_2 > 0$ } Positive

negative { $\lambda_1 < 0$, $\lambda_2 < 0$

SADDLE POINT

UNSTABLE



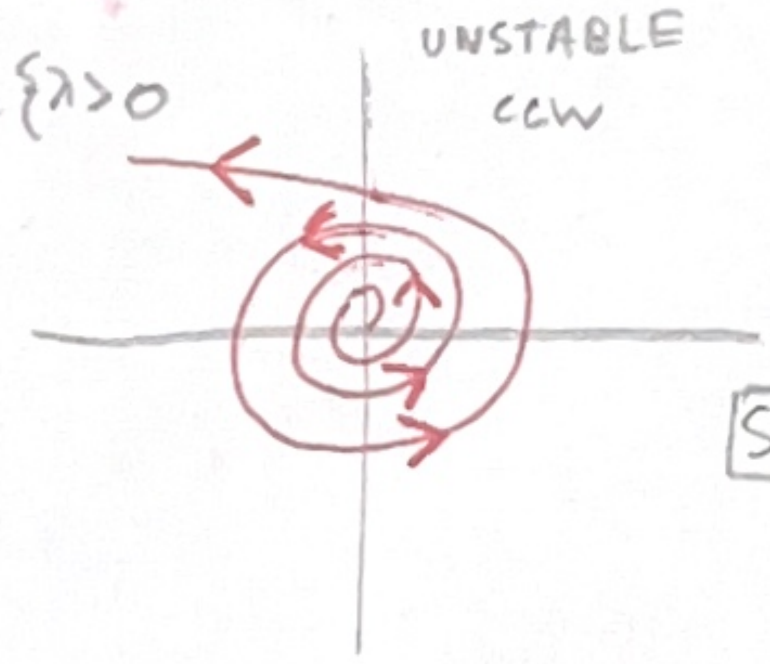
λ_1, λ_2 have opposite signs

Case 3:
Complex Eigenvalues

"When the real part is non-zero"

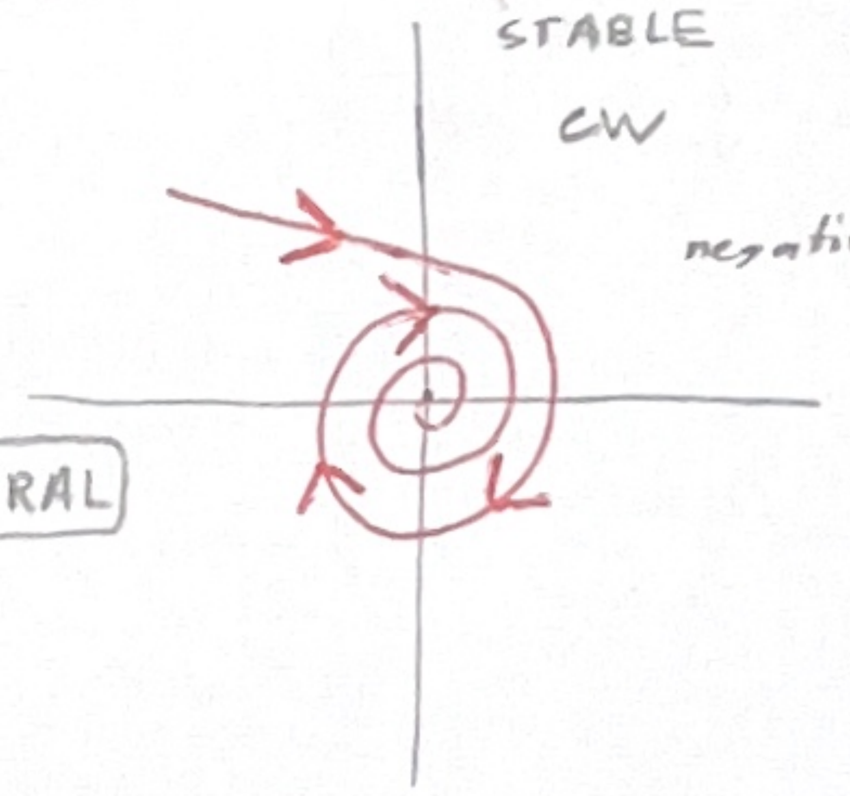
positive { $\lambda > 0$

UNSTABLE
ccw



STABLE
cw

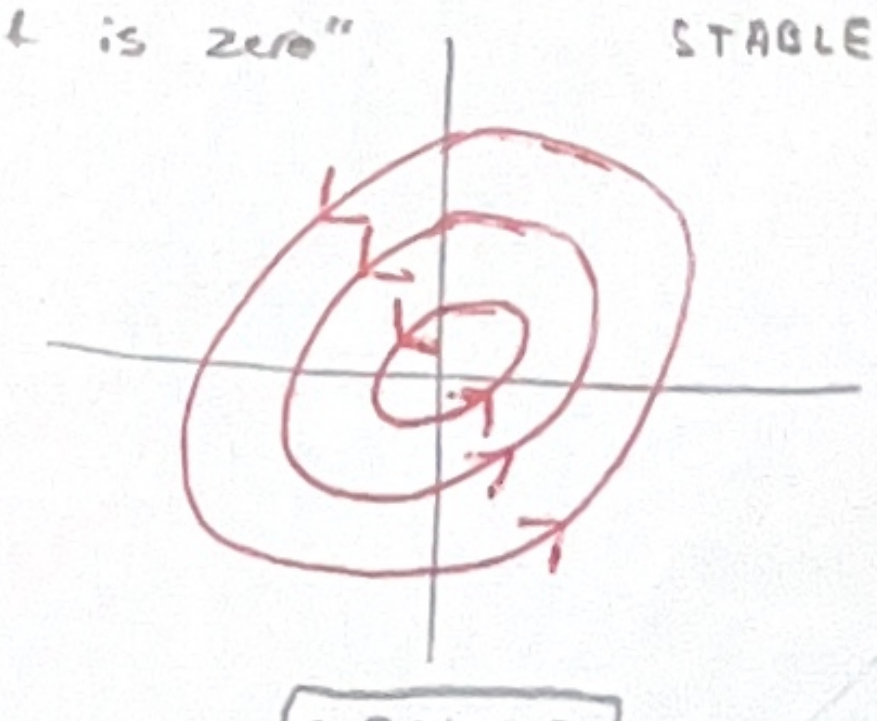
negative { $\lambda < 0$



SPIRAL

"When the real part is zero"

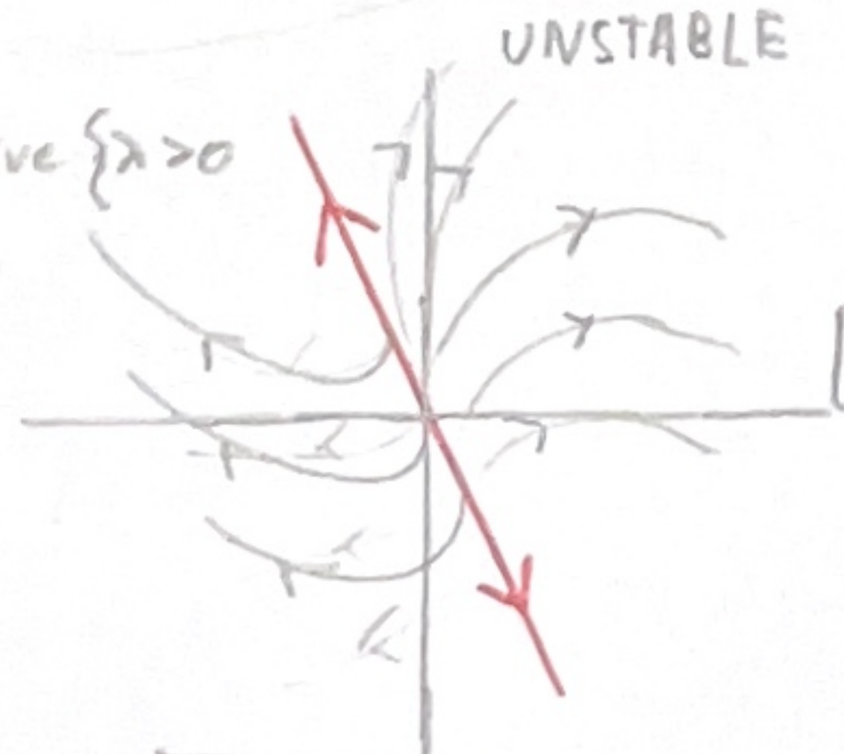
STABLE



CENTER

UNSTABLE

positive { $\lambda > 0$

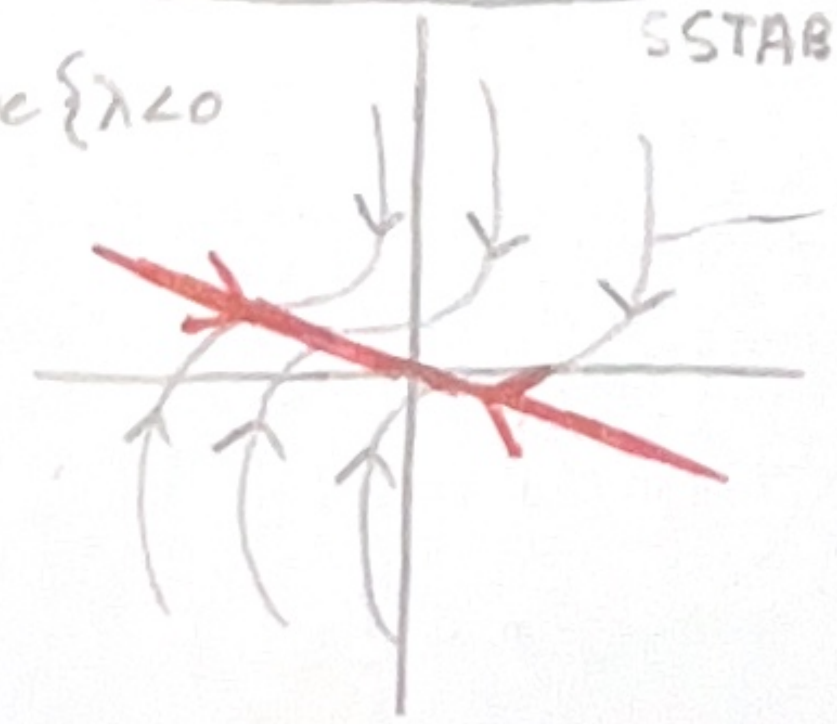


Case 2:
Repeated Eigenvalues

IMPROPER NODE

STABLE

negative { $\lambda < 0$



Boundary Value Problems

$y'' + p(x)y' + q(x)y = g(x)$

Homogenous when $g(x) = 0$

AND $y_0 = 0$ and $y_1 = 0$.

Otherwise → Inhomogenous

Types:

$y(x_0) = y_0 \quad y(x_1) = y_1$

$y'(x_0) = y_0 \quad y'(x_1) = y_1$

$y'(x_0) = y_0 \quad y(x_1) = y_1$

$y(x_0) = y_0 \quad y'(x_1) = y_1$

Periodic/orthogonal Functions

Periodic → $f(x+T) = f(x)$ for all x

- $\sin(wx)$ and $\cos(wx)$ are both periodic functions with $T = \frac{2\pi}{w}$

even → $f(-x) = f(x)$ ex: x^2, $\cos(x)$ $\quad \boxed{\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx}$

odd → $f(-x) = -f(x)$ ex: x^3, $\sin(x)$ $\quad \boxed{\int_{-L}^{L} f(x)\,dx = 0}$

Orthogonal

$f(x)$ and $g(x)$ are orthogonal on $a \le x \le b$ if

$$\int_a^b f(x)g(x)\,dx = 0$$

Extend ODD/EVEN

$f(x)$ is defined on $(0, L)$

$$f_{even}(x) = \begin{cases} f(-x), & -L < x < 0 \\ f(x), & 0 < x < L \end{cases}$$

$$f_{odd}(x) = \begin{cases} -f(-x), & -L < x < 0 \\ f(x), & 0 < x < L \end{cases}$$

Ex: $y'' + 4y = 0$; $y(0) = -2$, $y(2\pi) = -2$

$\therefore y(x) = C_1\cos(2x) + C_2\sin(2x)$

$\boxed{-2 = C_1}$

$\boxed{-2 = C_1}$

So.... $y(x) = -2\cos(2x) + C_2\sin(2x)$

C_2 can be anything so we have ∞ many solutions

Ex: $y'' + 4y = 0$; $y(0) = -2$, $y(2\pi) = 3$

$\therefore y(x) = C_1\cos(2x) + C_2\sin(2x)$

$\boxed{-2 = C_1}$

$\boxed{3 = C_1}$

So.... There are No solutions

Trig Identities

1. $\sin^2(x) + \cos^2(x) = 1$
2. $\sin(2x) = 2\sin(x)\cos(x)$
3. $\cos(2x) = \cos^2(x) - \sin^2(x)$
4. $\sin(a)\cos(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
5. $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
6. $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

Laplace Transformations

Definition

f(t) is a piecewise continuous function

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Linearity

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$$

Table of Common Laplace Transformations:

$f(t) = L^{-1}[F(s)]$	$Y(s) = L[f(t)]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$H(t-c)$ HEAVISIDE	$\frac{e^{-cs}}{s}$
$\delta(t-c)$ Dirac/Delta	e^{-cs}
$H(t-c)f(t-c)$	$e^{-cs} L[f(t)]$
$y'(t)$	$sY(s) - y(0)$
$y''(t)$	$s^2 Y(s) - sy(0) - y'(0)$
t^n *$n = 1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
t	$\frac{1}{s^2}$
$t^n f(t)$ *$n = 1,2,3\ldots$	$(-1)^n F^{(n)}(s)$

Partial Differential Equations

Steady State problem

$k u^{0}_{xx} + f(x) = 0$

Heat Equation

$u_t = k u_{xx}$ $\boxed{\text{Dirchlet}}$ $\longrightarrow$ $u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$

where $a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

$u_t = k u_{xx}$ $\boxed{\text{Neumann}}$ $\longrightarrow$ $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)$

$u_x(0,t) = u_x(L,t) = 0$

$u(x,0) = f(x)$

$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$

$u_t = k^2 u_{xx}$ $\boxed{\text{Periodic}}$ $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right)$

$u(-L,t) = u_x(L,t)$, $u_x(-L,t) = u(L,t)$

$u(x,0) = f(x)$

Where a_n, b_n are from Formula sheet

Wave Equation

$u_{tt} = ac^2 u_{xx}$ $\boxed{\text{Dirchlet}}$ $u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}ct\right) + b_n\sin\left(\frac{n\pi}{L}ct\right)\right)\sin\left(\frac{n\pi}{L}x\right)$

$u(0,t) = u(0,L) = 0$

$u(x,0) = f(x)$ $u_t(x,0) = g(x)$

$u_{tt} = c^2 u_{xx}$ $\boxed{\text{Neumann}}$ $u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}ct\right) + b_n\sin\left(\frac{n\pi}{L}ct\right)\right)\cos\left(\frac{n\pi}{L}x\right)$

$u_x(0,t) = u_x(L,t) = 0$

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$